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A Riemannian geometrical description for Lie–Poisson systems and its application to idealized magnetohydrodynamics

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Abstract. This paper presents a general theorem which enables description of Lie–Poisson systems for semi-direct product groups in terms of Riemannian geometry. The method employed is unique in that it changes the *non-quadratic* right- or left-invariant Hamiltonian to the *quadratic* form in a right- or left-invariant 1-form on the corresponding group by removing the respective invariance, which obtains the Riemannian metric and its induced Riemannian (Levi-Civita) connection. The resultant geodesic equation proves to be equivalent to the equation of motion, while the corresponding Jacobi equation determines its instability. In addition, this method is applied to idealized magnetohydrodynamics having isentropic flow, with a simple example being provided that considers the motion of an isentropic gas with no magnetic field present.

1. Introduction

This paper shows how Lie–Poisson systems for semi-direct product groups can be described in terms of Riemannian geometry, and how such a description can be applied to idealized magnetohydrodynamics (MHD) when considering only non-dissipative or isentropic flows. The method employed uses knowledge of the geodesics on finite- and infinite-dimensional Lie groups to study the instability of the motion in these systems, and is unique in that it can be used to obtain a Riemannian metric that changes the *non-quadratic* right- or left-invariant Hamiltonian to a *quadratic* form by removing the respective invariance.

Let $\mathcal{D}_\nu(M)$ be the Lie group of the C^∞ volume-preserving diffeomorphisms of a compact oriented N-manifold M which preserve the volume element $\nu_x = \sqrt{\det g_{jk}(x)}\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_N$ at each $x \in M$ for a local coordinate $(x_1, x_2, \ldots, x_N)$ near $x \in M$, where $g_{ij}(x)$ is the metric tensor on M for $x \in M$. Arnold (1966) showed that the motion of a rigid body and that of an ideal homogeneous incompressible fluid (I-fluid) on M can be represented as the geodesics on $SO(3)$ and on $\mathcal{D}_\nu(M)$, respectively. Ebin and Marsden (1970) subsequently developed a detailed functional analytic treatment of Arnold's approach by considering the ambient Lie group $\mathcal{D}(M)$ of all C^∞ diffeomorphisms of M (for further study see Bao *et al* (1993), and for an alternative formulation see Ono (1994)). Arnold (1966) additionally presented a theorem declaring that the right- or left-invariant metric obtained from a *quadratic* left- or right-invariant Hamiltonian makes the equations of motion determined by this Hamiltonian equivalent to the geodesic equation on the corresponding group.

By generalizing Arnold's theorem such that it allows the metrics obtained from the Hamiltonians to violate the right-/left-invariant property, the theorem is shown to include *non-quadratic* right-/left-invariant Hamiltonians. This new theorem is able to prove that the equation of motion for a Lie–Poisson system is equivalent to that of a geodesic on a certain group (see Marsden and Weinstein (1983) and Marsden *et al* (1984) for the Hamiltonian structures of Lie–Poisson systems). Thus, by means of the resultant sectional curvatures, we can investigate the instability of the system (for a completely different approach using symplectic connections, see Marsden *et al* (1991)). It should be noted that an example of the theorem presented was shown by Marsden (1976) for the motion of a *non-homogeneous* I-fluid in which a non-invariant metric was employed to obtain its motion as the geodesic on $\mathcal{D}_\upsilon(M)$.

Next, MHD fluid motion is shown to be the geodesic on a semi-direct product group $\mathcal{I}_*(M)$ of $\mathcal{D}(M)$, that is the group of all C^∞ diffeomorphisms of M, with the space $\Lambda^1(M)$ of 1-forms on M and the space $\mathcal{F}(M)$ of C^∞ functions on M, i.e.

$$\mathcal{I}_*(M) = \mathcal{D}(M) \times \Lambda^1(M) \times \mathcal{F}(M)$$

hence leading to the corresponding curvature. In addition, a simple example is given that considers the motion of an isentropic gas with no magnetic field present.

2. General method

2.1. Formulation

Let $\mathcal{G}$ be a finite- or infinite-dimensional Lie group and g be the Lie algebra of $\mathcal{G}$, while $\rho : \mathcal{G} \to \mathrm{Aut}(\mathcal{V})$ and $\rho_\perp : \mathcal{G} \to \mathrm{Aut}(\mathcal{V}_\perp)$ are the left Lie group representations of $\mathcal{G}$ in vector spaces $\mathcal{V}$ and $\mathcal{V}_\perp$ and $\rho' : g \to \mathrm{End}(V)$ and $\rho'_\perp : g \to \mathrm{End}(V_\perp)$ are the induced Lie algebra representations. Now, we define the semi-direct product $\mathcal{S}$ of $\mathcal{G}$ with $\mathcal{V}$ and $\mathcal{V}_\perp$ by introducing the following operation $*$ for $\Phi_1 = (\phi_1, \sigma_1, \sigma_1^\perp)$, $\Phi_2 = (\phi_2, \sigma_2, \sigma_2^\perp) \in \mathcal{S}$, i.e.

$$\Phi_1 * \Phi_2 = (\phi_1\phi_2, \sigma_1 + \rho(\phi_1)\sigma_2, \sigma_1^\perp + \rho_\perp(\phi_1)\sigma_2^\perp). \tag{2.1}$$

Letting s be the Lie algebra of $\mathcal{S}$, then the Lie bracket on s is defined for $V_1 = (v_1, \omega_1, \omega_1^\perp)$, $V_2 = (v_2, \omega_2, \omega_2^\perp) \in \mathcal{S}$ as

$$[V_1, V_2] = ([v_1, v_2], \rho'(v_1)\omega_2 - \rho'(v_2)\omega_1, \rho'_\perp(v_1)\omega_2^\perp - \rho'_\perp(v_2)\omega_1^\perp). \tag{2.2}$$

For a function $F(\Phi) \in \mathbb{R}$ of $\Phi \in \mathcal{G}$, the following two types of derivatives can be obtained:

$$V|_\Phi^+ F(\Phi) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} F(\Phi * \mathrm{e}^{\tau V}) \tag{2.3}$$

$$V|_\Phi^- F(\Phi) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} F(\mathrm{e}^{\tau V} * \Phi) \tag{2.4}$$

where $V|_\Phi^+$, $V|_\Phi^- \in T_\Psi\mathcal{S}$ in (2.3) and (2.4) represents the left- and right-invariant vectors at $\Phi \in \mathcal{S}$, respectively. The terms of $V|_\Phi^+$ or $V|_\Phi^-$ will be abbreviated as $V|_\Phi^\pm$. In the subsequent formulation, the abbreviation $\pm$ represents $+$ and $-$ signs when the left- and right-invariant fields (respectively) are considered. Thus, the relation between the commutation bracket of left- and right-invariant vectors and the Lie bracket of the corresponding elements of the Lie algebra can be obtained as $[V_1|_\Phi^\pm, V_2|_\Phi^\pm] = \pm[V_1, V_2]|_\Phi^\pm$ and $[V_1|_\Phi^+, V_2|_\Phi^-] = 0$.

If $\langle \mu, V\rangle$ denotes the natural pairing between $V \in s$ and $\mu \in s^*$, being the dual space of s, the left- or right-invariant 1-form $\mu|^{\pm}_{\Phi} \in T^*_{\Phi}\mathcal{S}$ corresponding to $\mu \in s^*$ can be defined by introducing the following natural pairing with $V|^{\pm}_{\Phi} \in T_{\Phi}\mathcal{S}$:

$$\langle \mu|^{\pm}_{\Phi}, V|^{\pm}_{\Phi}\rangle = \langle \mu, V\rangle. \tag{2.5}$$

Let $s_\star$ be the Lie algebra of the semi-direct product group $\mathcal{S}_\star = \mathcal{G} \times \mathcal{V}$, which can be considered as a subgroup of $\mathcal{S}$, and $s^*_\star$ be the dual space of $s_\star$. Also let a linear operator $g(\bar{\chi}, \bar{\chi}^\perp) : s_\star \to s^*_\star$ define a non-degenerate Euclidean structure on $s_\star$, i.e. for $(0, \bar{\chi}, \bar{\chi}^\perp) \in s^*$ and for $V_1, V_2 \in s_\star$,

$$\langle g(\bar{\chi}, \bar{\chi}^\perp)V_1, V_2\rangle = \langle g(\bar{\chi}, \bar{\chi}^\perp)V_2, V_1\rangle \tag{2.6}$$

which denotes $g(\bar{\chi}, \bar{\chi}^\perp)V = (g_1(\bar{\chi}, \bar{\chi}^\perp)v, g_2(\bar{\chi}, \bar{\chi}^\perp)\chi) \in s^*$.

We now consider a curve $\tilde{C} : I \to T^*\mathcal{S}$ for an open interval $I \subset \mathbb{R}$, where $\tilde{C}(t) \in T^*\mathcal{S}$ for $t \in I$ is locally represented as $\tilde{C}(t) = (\Phi_t, \mu_t|^{\pm}_{\Phi_t})$ for $\Phi_t = (\phi_t, \sigma_t, \sigma_t^\perp) \in \mathcal{S}$ and $\mu_t = (m_t, \chi_t, \chi_t^\perp) \in s^*$. Some important dynamical systems can be expressed using the following right-/left-invariant Hamiltonian which is partly quadratic in $m_t \in g$:

$$H(\Phi_t, \mu_t|^{\pm}_{\Phi_t}) = \tfrac{1}{2}\langle (m_t, \chi_t), g(\chi_t, \chi_t^\perp)^{-1}(m_t, \chi_t))\rangle. \tag{2.7}$$

This Hamiltonian can be regarded as a function on s^*, i.e. $H(\mu_t) = H(\Phi_t, \mu_t|^{\pm}_{\Phi_t})$ by reducing the phase space $T^*\mathcal{S}$ to s^* (consult Marsden and Weinstein 1983, Marsden *et al* 1984).

If the derivative

$$\frac{\partial F}{\partial \mu}(\mu) = \left(\frac{\partial F}{\partial m}(m), \frac{\partial F}{\partial \chi}(\chi), \frac{\partial F}{\partial \chi^\perp}(\chi^\perp)\right) \in s$$

of a function $F : s^* \to \mathbb{R}$ for $\mu = (m, \chi, \chi_\perp)$ and $\nu \in s^*$ is expressed as

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} F(\mu + \tau\nu) = \left\langle \nu, \frac{\partial F}{\partial \mu}\right\rangle \tag{2.8}$$

the dynamical system defined by the Hamiltonian of (2.7) is described by the following Lie–Poisson equation:

$$\frac{\mathrm{d}F(\mu_t)}{\mathrm{d}t} = \pm\left\langle \mu_t, \left[\frac{\partial H}{\partial \mu}(\mu_t), \frac{\partial F}{\partial \mu}(\mu_t)\right]\right\rangle. \tag{2.9}$$

When $\omega, \omega_\perp$ are elements of vector spaces $\mathcal{V}, \mathcal{V}_\perp$, while $\chi, \chi_\perp$ are in the dual spaces $\mathcal{V}^*, \mathcal{V}^*_\perp$, then $\rho(\phi)^*$ and $\rho_\perp(\phi)^*$ is defined for $\phi \in \mathcal{G}$ by

$$\begin{aligned} \langle \rho(\phi)^*\chi, \omega\rangle &= \langle \chi, \rho(\phi)\omega\rangle \\ \langle \rho_\perp(\phi)^*\chi, \omega\rangle &= \langle \chi, \rho_\perp(\phi)\omega\rangle \end{aligned} \tag{2.10}$$

and $\rho'(v)^*, \rho'_\perp(v)^*$ is defined for $v \in g$ by

$$\begin{aligned} \langle \rho'(v)^*\chi, \omega\rangle &= \langle \chi, \rho'(v)\omega\rangle \\ \langle \rho'_\perp(v)^*\chi, \omega\rangle &= \langle \chi, \rho'_\perp(v)\omega\rangle. \end{aligned} \tag{2.11}$$

Using (2.10) and (2.11), the equation of motion on a co-adjoint orbit on s^* is obtained as follows:

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}t} = \pm\, \mathrm{ad}^*_{\partial H/\partial\mu}\, \mu_t \tag{2.12}$$

that is

$$\begin{aligned} \frac{\mathrm{d}m_t}{\mathrm{d}t} &= \pm\{\mathrm{ad}^*_{\partial H/\partial m}\, m_t - (\rho'_{\partial H/\partial\chi})^*\chi_t - (\rho'_{\perp\partial H/\partial\chi^\perp})^*\chi_t^\perp\} \\ \frac{\mathrm{d}\chi_t}{\mathrm{d}t} &= \pm\rho'\left(\frac{\partial H}{\partial m}\right)^*\chi_t \\ \frac{\mathrm{d}\chi_t^\perp}{\mathrm{d}t} &= \pm\rho'_\perp\left(\frac{\partial H}{\partial m}\right)^*\chi_t^\perp \end{aligned} \tag{2.13}$$

where $(\rho'_{\partial H/\partial\chi})^*$ and $(\rho'_{\perp\partial H/\partial\chi^\perp})^*$ are defined for $(v, 0, 0) \in s$ as

$$\begin{aligned} &\langle((\rho'_{\partial/\partial\chi})^*\chi_t, 0, 0), (v, 0, 0)\rangle = \left\langle(0, \chi_t, 0), \left(0, \rho'(v)\frac{\partial H}{\partial\chi}, 0\right)\right\rangle \\ &\langle((\rho'_{\perp\partial H^\pm/\partial\chi^\perp})^*\chi_t^\perp, 0, 0), (v, 0, 0)\rangle = \left\langle\left(0, 0, \chi_t^\perp\right), \left(0, 0, \rho'(v)\frac{\partial H^\pm}{\partial\chi^\perp}\right)\right\rangle. \end{aligned} \tag{2.14}$$

For $\phi_t \in G$ such that $\partial H/\partial m = \phi_t^{-1}\dot{\phi}_t$ or $= \dot{\phi}_t\phi_t^{-1}$, the second and third equations of (2.13) respectively have the following general solutions for the initial values $\mu_0 = (m_0, \chi_0, \chi_0^\perp) \in s^*$:

$$\begin{aligned} \chi_t &= \{\rho(\phi_t)^*\}^{\pm 1}\chi_0 \\ \chi_t^\perp &= \{\rho_\perp(\phi_t)^*\}^{\pm 1}\chi_0^\perp. \end{aligned} \tag{2.15}$$

To study this system defined by the Hamiltonian (2.7) in terms of Riemannian geometry, we first define $(0, \bar{\chi}_\Phi, \bar{\chi}_\Phi^\perp) \in s^*$, which is determined by $\phi \in \mathcal{G}$ for $\mu = (m_0, \chi_0, \chi_0^\perp) \in s^*$ in (2.15) as

$$\begin{aligned} \bar{\chi}_\phi &= \{\rho(\phi)^*\}^{\pm 1}\chi_0 \\ \bar{\chi}_\phi^\perp &= \{\rho(\phi)^*\}^{\pm 1}\chi_0^\perp \end{aligned} \tag{2.16}$$

which has the same form as (2.15). Thus, for $\phi = \phi_t \in \mathcal{G}$,

$$\begin{aligned} \chi_t &= \bar{\chi}_{\phi_t} \\ \chi_t^\perp &= \bar{\chi}_{\phi_t}^\perp. \end{aligned} \tag{2.17}$$

This enables (2.7) to be described for $g_{\Phi_t} = g(\bar{\chi}_{\phi_t}, \bar{\chi}_{\phi_t}^\perp)$ as follows:

$$\begin{aligned} H_*(\Phi_t, \mu_t|^{\pm}_{\Phi_t}) &= \langle(m_t, \chi_t), g(\bar{\chi}_{\phi_t}, \bar{\chi}_{\phi_t}^\perp)^{-1}(m_t, \chi_t)\rangle \\ &= \langle(m_t, \chi_t), g_{\Phi_t}^{-1}(m_t, \chi_t)\rangle \end{aligned} \tag{2.18}$$

which is quadratic in $\mu_t \in s_* \subset s$, though the right-/left-invariant property is removed. The quadratic Hamiltonian of (2.18) induces the Riemannian structure defined by the following metric on $\mathcal{S}_*$ for $V_1, V_2 \in s_*$:

$$\langle\!\langle V_1|^{\pm}_{\Phi}, V_2|^{\pm}_{\Phi}\rangle\!\rangle|^{\pm}_{\Phi} = \langle g_\Phi V_1, V_2\rangle. \tag{2.19}$$

Therefore, it is reasonable to expect that the geodesic equation defined by the Riemannian (Levi-Civita) connection induced by the metric of (2.19) is equivalent to the equation of motion (2.12).

Theorem. The geodesic equation on $\mathcal{S}_\star$ defined by the Riemannian (Levi-Civita) connection induced by the metric of (2.19) is equivalent to the equation of motion (2.12), iff the following initial condition is satisfied for $(m_t, \chi_t) = g_{\Phi_t} V_t \in s_\star^*$:

$$\chi_t = \bar{\chi}_{\phi_t} \qquad \text{at } t = 0. \tag{2.20}$$

Proof. Using the metric of (2.19), we can uniquely define the Riemannian connection $\tilde{\nabla}$ and the induced covariant derivative $\tilde{\nabla}_{V_1|^\pm_\Phi} V_2|^\pm_\Phi$ on $\mathcal{S}_\star$ for V_1, V_2, and $V_3 \in s_\star$ from the following conditions of the Riemannian connection:

$$\begin{aligned} 2\langle\!\langle V_3|^\pm_\Phi, \tilde{\nabla}_{V_1|^\pm_\Phi} V_2|^\pm_\Phi\rangle\!\rangle|^\pm_\Phi &= \langle\!\langle V_3|^\pm_\Phi, [V_1|^\pm_\Phi, V_2|^\pm_\Phi]\rangle\!\rangle|^\pm_\Phi + \langle\!\langle V_2|^\pm_\Phi, [V_3|^\pm_\Phi, V_1|^\pm_\Phi]\rangle\!\rangle|^\pm_\Phi \\ &\quad + \langle\!\langle V_1|^\pm_\Phi, [V_3|^\pm_\Phi, V_2|^\pm_\Phi]\rangle\!\rangle|^\pm_\Phi - V_3|^\pm_\Phi \langle\!\langle V_1|^\pm_\Phi, V_2|^\pm_\Phi\rangle\!\rangle|^\pm_\Phi \\ &\quad + V_2|^\pm_\Phi \langle\!\langle V_3|^\pm_\Phi, V_1|^\pm_\Phi\rangle\!\rangle|^\pm_\Phi + V_1|^\pm_\Phi \langle\!\langle V_3|^\pm_\Phi, V_2|^\pm_\Phi\rangle\!\rangle|^\pm_\Phi \end{aligned} \tag{2.21}$$

which can easily be proven when applying the conditions of no torsion and metricity.

For $g'_\Phi : s_\star \times s_\star \to s_\star^*$ defined by

$$V_3|^\pm_\Phi \langle\!\langle V_1|^\pm_\Phi, V_2|^\pm_\Psi\rangle\!\rangle|^\pm_\Phi = \langle g'_\Phi V_2 V_1, V_3\rangle \tag{2.22}$$

the covariant derivative at $T_\Phi \mathcal{S}_\star$ can be calculated as

$$\begin{aligned} \tilde{\nabla}_{V_1|^\pm_\Phi} V_2|^\pm_\Phi &= \tfrac{1}{2}\{\pm\{[V_1, V_2] - g_\Phi^{-1}(\mathrm{ad}_{V_1})^* g_\Phi V_2 - g_\Phi^{-1}(\mathrm{ad}_{V_2})^* g V_1\} - g_\Phi^{-1} g'_\Phi V_2 V_1 \\ &\quad + g_\Phi^{-1}\{g_\Phi^{-1} g'_\Phi V_2\}^* g V_1 + g_\Phi^{-1}\{g_\Phi^{-1} g'_\Phi V_1\}^* g V_2\}|^\pm_\Phi. \end{aligned} \tag{2.23}$$

We now consider a geodesic curve $\hat{\Phi} : I \to \mathcal{S}_\star$ such that $\hat{\Phi}(t) = \Phi_t = (\phi_t, \sigma_t) \in \mathcal{S}_\star$. For $V_t|^+_{\Phi_t} = \Phi_t^{-1} * \dot{\Phi}_t|^+_{\Phi_t}$ or $V_t|^-_{\Phi_t} = \dot{\Phi}_t * \Phi_t^{-1} \in s_\star$, the geodesic equation is

$$\tilde{\nabla}_{V_t|^\pm_{\Phi_t}} V_t|^\pm_{\Phi_t} = 0 \tag{2.24}$$

and can therefore be described as

$$\begin{aligned} \dot{V}_t &\pm \tfrac{1}{2}\{-g_\Phi^{-1}(\mathrm{ad}_{V_t})^* g_\Phi V_t - g_\Phi^{-1}(\mathrm{ad}_{V_t})^* g_\Phi V_t\} \\ &\quad + \tfrac{1}{2}\{-g_\Phi^{-1} g'_\Phi V_t V_t + g_\Phi^{-1}\{g_\Phi^{-1} g'_\Phi V_t\}^* g_\Phi V_t + g_\Phi^{-1}\{g_\Phi^{-1} g'_\Phi V_t\}^* g_\Phi V_t\} = 0. \end{aligned} \tag{2.25}$$

Using (2.16) and the Hamiltonian of (2.20), equation (2.25) can be described for $(m_t, \chi_t) = g_{\Phi_t} V_t$ and $\mu_t = (m_t, \chi_t, \bar{\chi}^\perp_{\phi_t}) \in s^*$ as

$$\dot{\mu}_t = \pm\{(\mathrm{ad}_{\partial H_\star/\partial\mu})^* \mu_t - ((\rho'_{\partial H_\star/\partial\bar{\chi}_\phi})^* \bar{\chi}_{\phi_t}, 0, 0)\} \tag{2.26}$$

that is

$$\begin{aligned} \frac{\mathrm{d}m_t}{\mathrm{d}t} &= \pm\{\mathrm{ad}^*_{\partial H_\star/\partial m} m_t - (\rho'_{\partial H_\star/\partial\chi})^* \chi_t - (\rho'_{\partial H_\star/\partial\bar{\chi}^\perp_\phi})^* \bar{\chi}^\perp_{\phi_t} - (\rho'_{\partial H_\star/\partial\bar{\chi}_\phi})^* \bar{\chi}_{\phi_t}\} \\ \frac{\mathrm{d}\chi_t}{\mathrm{d}t} &= \pm\left\{\rho'\left(\frac{\partial H_\star}{\partial m}\right)^* \chi_t\right\} \\ \frac{\mathrm{d}\bar{\chi}^\perp_{\phi_t}}{\mathrm{d}t} &= \pm\left\{\rho'_\perp\left(\frac{\partial H_\star}{\partial m}\right)^* \bar{\chi}^\perp_{\phi_t}\right\}. \end{aligned} \tag{2.27}$$

This is possible due to the following relation: for any $\tilde{V} = (\tilde{v}, \tilde{\omega}, \tilde{\omega}^{\perp}) \in s$,

$$
\begin{aligned}
\langle ((\rho'_{\partial H_*/\partial\bar{\chi}_\phi})^* \chi_\phi + (\rho'_{\partial H_*/\partial\bar{\chi}^{\perp}_\phi})^* \chi^{\perp}_\phi, 0, 0), \tilde{V}\rangle &= \left\langle (0, \rho'(\tilde{v})^* \chi_\phi, \rho'(\tilde{v})^* \chi^{\perp}_\phi), \left(0, \frac{\partial H_*}{\partial \bar{\chi}_\phi}, \frac{\partial H_*}{\partial \bar{\chi}^{\perp}_\phi}\right)\right\rangle \\
&= \pm \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} H_*(\bar{\chi}_{e^{\tau\tilde{v}}\phi}, \bar{\chi}^{\perp}_{e^{\tau\tilde{v}}\phi}) \\
&= \pm \tfrac{1}{2}\langle -g'_\Phi(g_\Phi^{-1}\mu)(g_\Phi^{-1}\mu), (\tilde{v}, \tilde{\omega})\rangle.
\end{aligned}
\tag{2.28}
$$

On the other hand, $\bar{\chi}_{\phi_t}$ obeys the following differential equation:

$$
\frac{\mathrm{d}\bar{\chi}_{\phi_t}}{\mathrm{d}t} = \pm \rho' \left(\frac{\partial H_*}{\partial m}\right)^* \bar{\chi}_{\phi_t} \tag{2.29}
$$

which has the same form as the second equation of (2.13). Thus, the initial condition of (2.20) results in $\chi_t = \bar{\chi}_{\phi_t}$, for $t \geqslant 0$. Therefore, for $\mu_t = (m_t, \chi_t, \chi_t^{\perp}) \in s^*$, (2.26) becomes

$$
\frac{\mathrm{d}\mu_t}{\mathrm{d}t} = \pm\, \mathrm{ad}^*_{\partial H/\partial\mu}\, \mu_t \tag{2.30}
$$

because

$$
\begin{aligned}
\frac{\partial H}{\partial m} &= \frac{\partial H_*}{\partial m} \\
\frac{\partial H}{\partial \chi} &= \left.\left(\frac{\partial H_*}{\partial \chi} + \frac{\partial H_*}{\partial \bar{\chi}}\right)\right|_{\chi_t = \bar{\chi}_{\phi_t}} \\
\frac{\partial H}{\partial \chi^{\perp}} &= \left.\frac{\partial H_*}{\partial \bar{\chi}^{\perp}}\right|_{\chi_t^{\perp} = \bar{\chi}^{\perp}_{\phi_t}}.
\end{aligned}
\tag{2.31}
$$

□

2.2. Prediction of instability

The Riemannian structure on $\mathcal{S}_*$ presents one criterion for instability of the system. For a parameter $\varepsilon \in (-1, 1)$ and for $W_t = (w_t, \varsigma_t, \varsigma_t^{\perp}) \in s_*$, $\Psi^{\pm}_{t,\varepsilon} \in \mathcal{S}_*$ can be defined as

$$
\Psi^{+}_{t,\varepsilon} = \Phi_t * e^{\varepsilon W_t} \tag{2.32}
$$

or

$$
\Psi^{-}_{t,\varepsilon} = e^{\varepsilon W_t} * \Phi_t \tag{2.33}
$$

and the right-invariant vectors $V_t^{\varepsilon}|^{\pm}_{\Psi_{t,\varepsilon}} = (v_t^{\varepsilon}, \omega_t^{\varepsilon}, \omega_t^{\perp\varepsilon})|^{\pm}_{\Psi_{t,\varepsilon}}$, $W_t|^{\pm}_{\Psi_{t,\varepsilon}} \in T_{\Psi_{t,\varepsilon}}\mathcal{S}_*$ are obtained as

$$
\begin{aligned}
W_t|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}} F(\Psi^{\pm}_{t,\varepsilon}) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\Psi^{\pm}_{t,\varepsilon}) \\
V_t^{\varepsilon}|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}} F(\Psi^{\pm}_{t,\varepsilon}) &= \frac{\mathrm{d}}{\mathrm{d}t} F(\Psi^{\pm}_{t,\varepsilon})
\end{aligned}
\tag{2.34}
$$

which satisfy

$$V_t^\varepsilon|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}} = V_t|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}} + \varepsilon[V_t|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}}, W_t|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}}] \tag{2.35}$$

$$\Leftrightarrow V_t^\varepsilon = V_t + \varepsilon(\dot{W}_t \pm \mathrm{ad}_{V_t} W_t) \tag{2.36}$$

since for $W_t^0 = \{\mathrm{Ad}_{\Phi_t}\}^{\pm 1} W_t \in s_*$ such that $\Psi^+_{t,\varepsilon} = \mathrm{e}^{\varepsilon W_t^0} * \Phi_t$ or $\Psi^-_{t,\varepsilon} = \Phi_t * \mathrm{e}^{\varepsilon W_t^0}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\Psi^{\pm}_{t,\varepsilon}) = (V_t + \varepsilon\{\mathrm{Ad}_{\Phi_t}\}^{-(\pm 1)} \dot{W}_t^0)|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}} F(\Psi^{\pm}_{t,\varepsilon}) \tag{2.37}$$

and

$$\begin{aligned}\{\mathrm{Ad}_{\Phi_t}\}^{-(\pm 1)} \dot{W}_t^0|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}} &= \dot{W}_t \pm \mathrm{ad}_{V_t} W_t|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}} \\ &= [V_t|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}}, W_t|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}}].\end{aligned} \tag{2.38}$$

In particular, the vector field constructed by vectors $W_t|^{\pm}_{\Phi_t} = W_t|^{\pm}_{\Psi^{\pm}_{t,0}}$ on $\mathcal{S}_*$ is called a Jacobi field. Letting $\Psi^{\pm}_{t,\varepsilon} \in \mathcal{S}_*$ represent a geodesic on $\mathcal{S}$, we obtain

$$\tilde{\nabla}_{V_t^\varepsilon|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}}} V_t^\varepsilon|^{\pm}_{\Psi^{\pm}_{t,\varepsilon}} = 0. \tag{2.39}$$

If (2.39) is the equation of motion, the following initial condition for $(m_t^\varepsilon, \chi_t^\varepsilon, \chi_t^{\perp\varepsilon}) = g_{\Psi^{\pm}_{t,\varepsilon}} V_t^\varepsilon$ must be satisfied:

$$\chi_t^\varepsilon = \bar{\chi}_{\Psi^{\pm}_{t,\varepsilon}} \qquad \text{at } t = 0 \tag{2.40}$$

which holds not only at $t = 0$ but also for $t > 0$. This is true since χ_t^ε and $\chi_t^{\perp\varepsilon}$ respectively satisfy the same differential equations as $\bar{\chi}_{\Psi^{\pm}_{t,\varepsilon}}$ and $\bar{\chi}^{\perp}_{\Psi^{\pm}_{t,\varepsilon}}$ based on their definitions in (2.15) and (2.16). Thus, the following condition must be satisfied:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \chi_t^\varepsilon = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \bar{\chi}_{\Psi^{\pm}_{t,\varepsilon}}. \tag{2.41}$$

Differentiating (2.39) by ε at $\varepsilon = 0$ enables us to obtain the Jacobi equation for the curvature $\tilde{R}(V_1|^{\pm}_{\Psi}, V_2|^{\pm}_{\Psi}) = [\tilde{\nabla}_{V_1|^{\pm}_{\Psi}}, \tilde{\nabla}_{V_2|^{\pm}_{\Psi}}] - \tilde{\nabla}_{[V_1|^{\pm}_{\Psi}, V_2|^{\pm}_{\Psi}]}$, i.e.

$$\tilde{\nabla}_{V_t|^{\pm}_{\Phi_t}} \tilde{\nabla}_{V_t|^{\pm}_{\Phi_t}} W_t|^{\pm}_{\Phi_t} = -\tilde{R}(W_t|^{\pm}_{\Phi_t}, V_t|^{\pm}_{\Phi_t}) V_t|^{\pm}_{\Phi_t} \tag{2.42}$$

from which we can easily obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\{\langle\!\langle W_t|^{\pm}_{\Phi_t}, W_t|^{\pm}_{\Phi_t}\rangle\!\rangle|^{\pm}_{\Phi_t}\} = 2\{&\langle\!\langle \tilde{\nabla}_{V_t|^{\pm}_{\Phi_t}} W_t|^{\pm}_{\Phi_t}, \tilde{\nabla}_{V_t|^{\pm}_{\Phi_t}} W_t|^{\pm}_{\Phi_t}\rangle\!\rangle|^{\pm}_{\Phi_t} \\ &- \langle\!\langle W_t|^{\pm}_{\Phi_t}, \tilde{R}(W_t|^{\pm}_{\Phi_t}, V_t|^{\pm}_{\Phi_t}) V_t|^{\pm}_{\Phi_t}\rangle\!\rangle|^{\pm}_{\Phi_t}\}.\end{aligned} \tag{2.43}$$

This equation indicates that if the sectional curvature $\langle\!\langle W_t|^{\pm}_{\Phi_t}, \tilde{R}(W_t|^{\pm}_{\Phi_t}, V_t|^{\pm}_{\Phi_t}) V_t|^{\pm}_{\Phi_t}\rangle\!\rangle^{\pm}_{\Phi_t}$ is negative, then the sign of (2.43) must be positive, that is, the system is unstable in this sense.

2.3. Classification

The mechanics described by a Hamiltonian such as (2.8) can be classified as follows:

(i) $\mathcal{G} = \mathcal{S}_* = \mathcal{S}$

e.g. the motion of a rigid body:

$$\mathcal{G} = \mathcal{S}_* = \mathcal{S} = \mathrm{SO}(3)$$

and the motion of a homogeneous I-fluid:

$$\mathcal{G} = \mathcal{S}_* = \mathcal{S} = \mathcal{D}_\nu(M).$$

(ii) $\mathcal{G} = \mathcal{S}_* \subsetneqq \mathcal{S}$

e.g. the motion of a non-homogeneous I-fluid:

$$\mathcal{G} = \mathcal{S}_* = \mathcal{D}_\nu(M) \qquad \mathcal{S} = \mathcal{D}_\nu(M) \times \mathcal{F}(M).$$

(iii) $\mathcal{G} \subsetneqq \mathcal{S}_* = \mathcal{S}$

e.g. the motion of a heavy top (a top under gravity):

$$\mathcal{G} = \mathrm{SO}(3) \qquad \mathcal{S}_* = \mathcal{S} = \mathrm{SO}(3) \times E^3$$

and the motion of a homogeneous I-MHD fluid:

$$\mathcal{G} = \mathcal{D}_\nu(M) \qquad \mathcal{S}_* = \mathcal{S} = \mathcal{D}_\nu(M) \times \Lambda^1(M).$$

(iv) $\mathcal{G} \subsetneqq \mathcal{S}_* \subsetneqq \mathcal{S}$

e.g. the motion of a non-homogeneous I-MHD fluid:

$$\mathcal{G} = \mathcal{D}_\nu(M) \qquad \mathcal{S}_* = \mathcal{D}_\nu(M) \times \Lambda^1(M) \qquad \mathcal{S} = \mathcal{D}_\nu(M) \times \Lambda^1(M) \times \mathcal{F}(M)$$

and the motion of an isentropic fluid:

$$\mathcal{G} = \mathcal{D}(M) \qquad \mathcal{S}_* = \mathcal{D}(M) \times \mathcal{F}(M) \qquad \mathcal{S} = \mathcal{D}_\nu(M) \times \mathcal{F}(M) \times \mathcal{F}(M)$$

and the motion of an isentropic MHD fluid:

$$\mathcal{G} = \mathcal{D}(M) \qquad \mathcal{S}_* = \mathcal{D}(M) \times \Lambda^1(M) \times \mathcal{F}(M)$$
$$\mathcal{S} = \mathcal{D}_\nu(M) \times \Lambda^1(M) \times \mathcal{F}(M) \times \mathcal{F}(M).$$

3. MHD fluid motion

3.1. Formulation

First, we apply the method in section 2 to the motion of an isentropic MHD fluid on a 3-manifold M, in which the corresponding Lie group is a semidirect product of $\mathcal{D}(M)$ with $\Lambda^1(M)$, $\mathcal{F}(M)$ (the space of C^∞ functions on M), and one more $\mathcal{F}(M)$, i.e.

$$\mathcal{I}(M) = \mathcal{D}(M) \times \Lambda^1(M) \times \mathcal{F}(M) \times \mathcal{F}(M). \tag{3.1}$$

For $\Psi_1 = (\psi_1, h_1, f_1, g_1), \Psi_2 = (\psi_2, h_2, f_2, g_2) \in \mathcal{I}(M)$, the product of two elements of $\mathcal{I}(M)$ is defined as follows:

$$\begin{aligned}\Psi_1 * \Psi_2 &= (\psi_1, h_1, f_1, g_1) * (\psi_2, h_2, f_2, g_1)\\ &= (\psi_1 \circ \psi_2, \psi_2^* h_1 + h_2, \psi_2^* f_1 + f_2, \psi_2^* g_1 + g_2)\end{aligned} \tag{3.2}$$

where ψ^* denotes the pull-back by $\psi \in \mathcal{D}(M)$ and the unit element of $\mathcal{I}(M)$ can be denoted as $(e, 0, 0, 0) \in \mathcal{I}(M)$, where $e \in \mathcal{D}(M)$ corresponds to the identity map from M to itself. Moreover, for a function $F : \mathcal{I}(M) \to \mathbb{R}$, the right-invariant vector $V|_\Psi^-$ for $\Psi \in \mathcal{I}(M)$ can be defined as

$$(V|_\Psi^-)F(\Psi) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} F(\mathrm{e}^{\tau V} * \Psi) \tag{3.3}$$

where V is an element of $i(M)$, being the Lie algebra of $\mathcal{I}(M)$. For $V_1, V_2 \in i(M)$, the commutation bracket becomes $[V_1|_\Psi^-, V_2|_\Psi^-] = -[V_1, V_2]|_\Psi^-$. Thus, a right-invariant vector $V|_{(e,0,0,0)}^-$ at $T_{(e,0,0,0)}\mathcal{I}(M)$ can be identified with the corresponding element of $i_+(M)$, being the right Lie algebra of $\mathcal{I}(M)$, with a Lie bracket that is the negative of that of $i(M)$, and also with the corresponding set of a vector field, a function, one more function, and a 1-form on M, i.e.

$$V|_{(e,0,0,0)}^- = (v, H, U, W)|_{(e,0,0,0)}^- = (H_i(x)\,\mathrm{d}x^i, v^i(x)\partial_i, U(x), W(x)) \tag{3.4}$$

where ∂_i and $\mathrm{d}x^i$ are, respectively, the bases of T_xM and T_x^*M. Thus, a right-invariant vector $V|_\Psi^- \in T_\Psi\mathcal{I}(M))$ obtained by the right translation of $V|_{(e,0,0,0)}^-$ by $\Psi \in \mathcal{I}(M)$ can be represented as

$$V|_\Psi^- = (v, U, W, H)|_\Psi^- = (v^i(x)\partial_i, H_i(x)\,\mathrm{d}x^i, U(x), W(x))|_\Psi^-. \tag{3.5}$$

The commutation bracket for $V_1|_\Psi^- = (v_1^i(x)\partial_i, H_{1i}(x)\,\mathrm{d}x^i, U_1(x), W_1(x))|_\Psi^-$ and $V_2|_\Psi^- = (v_2^i(x)\partial_i, H_{2i}(x)\,\mathrm{d}x^i, U_2(x), W_2(x))|_\Psi^- \in T_\Psi\mathcal{I}(M)$ becomes

$$\begin{aligned}[V_1|_\Psi^-, V_2|_\Psi^-] = ([v_1^i\partial_i, v_2^j\partial_j], &\mathrm{L}_{v_1^i\partial_i} H_{j2}\,\mathrm{d}x^j - \mathrm{L}_{v_2^j\partial_j} H_{i1}\,\mathrm{d}x^i, v_1^j\partial_j U_2(x)\\ &- v_2^j\partial_j U_1(x), v_1^j\partial_j W_2(x) - v_2^j\partial_j W_1(x))|_\Psi^-\end{aligned} \tag{3.6}$$

where $\mathrm{L}_{v^j\partial_j}$ denotes the Lie derivative by $v^j\partial_j$.

Introducing a natural pairing defines an element of $T_\Psi^*\mathcal{I}(M)$, which can be denoted for $\Psi \in \mathcal{I}(M)$ as

$$\mu|_\Psi^- = (m, B, \rho, \sigma)|_\Psi^- = (\nu_x \otimes m_j(x)\,\mathrm{d}x^j, B_{ij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j, \nu_x\rho(x), \nu_x\sigma(x))|_\Psi^- \tag{3.7}$$

equation (3.7) physically means the set of momentum density, magnetic induction field, mass density and entropy density.

We now consider a curve $\tilde{C} : I \to T^*\mathcal{I}(M)$, where $\tilde{C}(t) \in T^*\mathcal{I}(M)$ for $t \in I$ is locally represented as $\tilde{C}(t) = (\Psi_t, \mu_t|_{\Psi_t}^-) \in T^*\mathcal{I}(M)$. For the thermodynamic internal energy $U(\rho(x), \sigma(x))$, the following *non-quadratic* right-invariant Hamiltonian defines the motion of an isentropic MHD fluid (see Marsden *et al* 1984):

$$\begin{aligned}H(\Psi_t, \mu_t|_{\Psi_t}^-) = \tfrac{1}{2}\bigg\{ &\int_M \nu_x\rho_t(x)^{-1} g^{ij}(x) m_{ti}(x) m_{tj}(x)\\ &+ \int_M B_{tij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge {}^*(B_{tij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j)\\ &+ 2\int_M \nu_x\{\rho_t(x)^{-1} U(\rho_t(x), \sigma_t(x))\}\rho_t(x)\rho_t(x)\bigg\}\end{aligned} \tag{3.8}$$

where $g_{ij}(x)g^{jk}(x) = \delta_i^k$ (: $\delta_i^k = 1$ for $i = j$, $\delta_i^k = 0$ for $i \neq j$) and the magnetic permeability is taken to be unity. As a result, the equation of motion for this fluid can be obtained as the equation of motion on the co-adjoint orbit on $i^*(M)$.

To obtain a *quadratic* Hamiltonian instead of (3.8) using the same method as in section 2.1, we consider a semidirect product group $\mathcal{I}_*(M)$ defined as

$$\mathcal{I}_*(M) = \Lambda^1(M) \times \mathcal{D}(M) \times \mathcal{F}(M) \tag{3.9}$$

which can be regarded as a subgroup of $\mathcal{I}(M)$ by identifying $\Psi = (\psi, h, f) \in \mathcal{I}_*(M)$ with $(\psi, h, f, 0) \in \mathcal{I}(M)$. A right-invariant vector $V|_{\Psi}^{-} \in T_\Psi \mathcal{I}_*(M)$ can be described as

$$V|_{\Psi}^{-} = (v, H, U)|_{\Psi}^{-} = (v^i(x)\partial_i, H_i(x)\,\mathrm{d}x^i, U(x))|_{\Psi}^{-}. \tag{3.10}$$

For the Jacobian $J_\psi(x) = \det(\partial\psi(x)^j/\partial x^i)$ at $\Psi = (\psi, h, f) \in T_\Psi\mathcal{I}_*(M)$, the mass density, entropy density and internal energy can be obtained as the functions on M, i.e.

$$\begin{aligned} &\textit{mass density:} && \bar{\rho}_\psi(x) = \rho_0 \circ \psi^{-1}(x) J_{\psi^{-1}}(x) \\ &\text{entropy density:} && \bar{\sigma}_\psi(x) = \sigma_0 \circ \psi^{-1}(x) J_{\psi^{-1}}(x) \\ &\text{internal energy:} && \bar{U}_\psi(x) = \bar{U}(\rho_\psi(x), \sigma_\psi(x)) \end{aligned} \tag{3.11}$$

where $\rho_0(x)$ and $\sigma_0(x)$ are, respectively, the initial mass density and initial entropy density, and internal energy only depends on $\rho_\psi(x)$ and $\sigma_\psi(x)$. For a function

$$\alpha_\psi(x) = \alpha(\bar{\rho}_\psi(x)) = \bar{\rho}_\psi(x)/\bar{U}_\psi(x)$$

on M and a right-invariant 1-form

$$\mu_t|_{\Psi_t}^{-} = (\nu_x \otimes m_{tj}(x)\,\mathrm{d}x^j, B_{tij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j, \nu_x\rho_t(x))|_{\Psi_t}^{-} \in T^*_{\Psi_t}\mathcal{S}(M)$$

at $\Psi_t \in \mathcal{I}_*(M)$, we can define the following Hamiltonian:

$$\begin{aligned} H(\Psi_t, \mu_t|_{\Psi_t}^{-}) = \tfrac{1}{2}\bigg\{ &\int_M \nu_x \rho_{\psi_t}(x)^{-1} g^{ij}(x) m_{ti}(x) m_{tj}(x) \\ &+ \int_M B_{tij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge {}^*(B_{tij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j) \\ &+ 2\int_M \nu_x \alpha_{\psi_t}(x)^{-1} \rho_t(x)\rho_t(x) \bigg\} \end{aligned} \tag{3.12}$$

which is quadratic in $\mu_t|_{\Psi_t}^{-} \in T_{\Psi_t}\mathcal{I}_*(M)$ though no longer right-invariant. If $\bar{\rho}_{\psi_t} = \rho_t$, then (3.12) has the same form as (3.8), and $m_{tj}(x)$, $B_{tij}(x)$, and $\rho_t(x)$ physically correspond to momentum density, magnetic induction field, and mass density, respectively.

For

$$\begin{aligned} V_1|_{\Psi}^{-} &= (v_1, H_1, U_1)|_{\Psi}^{-} = (v_1^i(x)\partial_i, H_{1i}(x)\,\mathrm{d}x^i, U_1(x))|_{\Psi}^{-} \\ V_2|_{\Psi}^{-} &= (v_2, H_2, U_2)|_{\Psi}^{-} = (v_2^i(x)\partial_i, H_{2i}(x)\,\mathrm{d}x^i, U_2(x))|_{\Psi}^{-} \in T_\Psi\mathcal{I}_*(M) \end{aligned}$$

we can endow $\mathcal{I}_*(M)$ with a Riemannian structure by introducing the following metric corresponding to (3.12):

$$\langle\!\langle V_1|_{\bar{\Psi}}, V_2|_{\bar{\Psi}}\rangle\!\rangle|_{\bar{\Psi}} = \int_M \rho_\psi(x)\nu_x g_{ij}(x)v_1^i(x)v_2^j(x) + \int_M {}^*(H_{1i}(x)\,\mathrm{d}x^i)\wedge H_{2j}(x)\,\mathrm{d}x^j$$
$$+ 2\int_M \nu_x\alpha_\psi(x)U_1(x)U_2(x) \tag{3.13}$$

which defines a one-to-one correspondence between a right-invariant 1-form

$$\mu|_{\bar{\Psi}} = (\nu_x\otimes m_j(x)\,\mathrm{d}x^j, B_{ij}(x)\,\mathrm{d}x^i\wedge\mathrm{d}x^j, \nu_x\rho(x))|_{\bar{\Psi}} \in T^*_\Psi\mathcal{I}_*(M)$$

and a right-invariant vector

$$V|_{\bar{\Psi}} = (v^j(x)\partial_j, H_i(x)\,\mathrm{d}x^i, U(x))|_{\bar{\Psi}} \in T_\Psi\mathcal{I}_*(M)$$

as

$$\mu|_{\bar{\Psi}} = (\nu_x\otimes m_j(x)\,\mathrm{d}x^j, B_{ij}(x)\,\mathrm{d}x^i\wedge\mathrm{d}x^j, \nu_x\tilde{\rho}(x))|_{\bar{\Psi}}$$
$$= (\nu_x\otimes\{\rho_\psi(x)g(x)_{jk}v^k(x)\}\,\mathrm{d}x^j, {}^*(H_i(x)\,\mathrm{d}x^i), \nu_x\{w_\psi(x)\tilde{U}(x)\})|_{\bar{\Psi}}. \tag{3.14}$$

Then, if $\bar{U}_\psi = U$ (or $\rho_\psi = \rho$), $V|_{\bar{\Psi}} \in T_\Psi\mathcal{I}_*(M)$ represents the set of velocity, internal energy and magnetic field.

Under the conditions of the Riemannian connection of (2.21), the metric of (3.13) uniquely defines the Riemannian connection $\tilde{\nabla}$ and the induced covariant derivative $\tilde{\nabla}_{V_1|_{\bar{\Psi}}}V_2|_{\bar{\Psi}}$ on $T_\Psi\mathcal{I}_*(M)$ for $V_1|_{\bar{\Psi}} = (v_1, H_1, U_1)|_{\bar{\Psi}}$, $V_2|_{\bar{\Psi}} = (v_2, H_2, U_2)|_{\bar{\Psi}}$, and $V_3|_{\bar{\Psi}} = (v_3, H_3, U_3)|_{\bar{\Psi}} \in T_\Psi\mathcal{I}_*(M)$. When the metric tensor on M is $g_{jk}(x) = \delta_{jk}$ (where δ_{jk} is Kronecker's delta) for any $x\in M$, the covariant derivatives on $T_\Psi\mathcal{I}_*(M)$ can be calculated for

$$\beta_\psi(x) = \alpha_\psi(x) - \left\{\rho_\psi(x)\frac{\partial\alpha_\psi}{\partial\rho_\psi}(x) + \sigma_\psi(x)\frac{\partial\alpha_\psi}{\partial\sigma_\psi}(x)\right\}$$

and

$$\gamma_\psi(x) = 2 - \alpha_\psi(x)^{-1}\left\{\rho_\psi(x)\frac{\partial\alpha_\psi}{\partial\rho_\psi}(x) + \sigma_\psi(x)\frac{\partial\alpha_\psi}{\partial\sigma_\psi}(x)\right\}$$

using Green's theorem, i.e.

$$\tilde{\nabla}_{V_1|_{\bar{\Psi}}}V_2|_{\bar{\Psi}} = \frac{1}{2}\Bigg(\Bigg\{2v_1^j(x)\partial_j v_2^i(x) + \frac{1}{\rho_\psi(x)}\partial^i(H_1^j(x)H_{2j}(x)) - \frac{1}{\rho_\psi(x)}\partial^j(H_{2j}(x)H_1^i(x))$$
$$- \frac{1}{\rho_\psi(x)}\partial^j(H_{1j}(x)H_2^i(x)) + \frac{2}{\rho_\psi(x)}\partial^i\{\beta_\psi U_1(x)U_2(x)\}\Bigg\}\partial_i,$$
$$\big\{v_1^j(x)\partial_j H_{i2}(x) + H_{j2}(x)\partial_i v_1^j(x) - v_2^j(x)\partial_j H_{i1}(x) - H_{j1}(x)\partial^i v_2^j(x)$$
$$- H_2^j(x)\partial_j v_{1i}(x) + \partial_j(v_1^j(x)H_{2i}(x))$$
$$- H_1^j(x)\partial_j v_{2i}(x) + \partial_j(v_2^j(x)H_{1i}(x))\big\}\,\mathrm{d}x^i,$$
$$2v_1^j\partial_j U_2 + (\gamma_\psi - 1)U_2\partial_j v_1^j + (\gamma_\psi - 1)U_1\partial_j v_2^j\Bigg)\Bigg|_{\bar{\Psi}}. \tag{3.15}$$

This calculation is made possible by taking the advantage of the fact that (3.3) and (3.13) give the following relation:

$$V_3|_{\bar\Psi}\langle\langle V_1|_{\bar\Psi}, V_2|_{\bar\Psi}\rangle\rangle|_{\bar\Psi} = -\int_M \nu_x \partial_j\{v_3^j \rho_\psi(x)\} g_{ij}(x) v_1^i(x) v_2^j(x)$$
$$-2\int_M \nu_x \left\{\rho_\psi(x)\frac{\partial\alpha_\psi}{\partial\rho_\psi}(x) + \sigma_\psi(x)\frac{\partial\alpha_\psi}{\partial\sigma_\psi}(x)\right\} U_1(x)U_2(x)\partial_j v_3^j$$
$$+2\int_M \nu_x \alpha_\psi(x)\partial_j\{v_3^j U_1(x)U_2(x)\}.$$

Now, let us consider a geodesic curve $\hat\Psi : \mathbb{R} \to \mathcal{I}_*(M)$ such that $\hat\Psi(t) = \Psi_t = (\psi_t, h, f_t) \in \mathcal{I}_*(M)$. For $V_t|_{\bar\Psi_t} = \dot\Psi_t * \Psi_t^{-1}|_{\bar\Psi_t} = (v_t, H_t, U_t)|_{\bar\Psi_t} \in T_\Psi\mathcal{I}_*(M)$, the geodesic equation is

$$\tilde\nabla_{V_t|_{\bar\Psi_t}} V_t|_{\bar\Psi_t} = 0 \tag{3.16}$$

and can therefore be described for $\bar\rho_t = \bar\rho_{\psi_t}$, $\bar\sigma_t = \bar\sigma_{\psi_t}$, $\alpha_t = \alpha_{\psi_t}$, $\beta_t = \beta_{\psi_t}$, and $\gamma_t = \gamma_{\psi_t}$ as

$$\begin{aligned}
&\frac{\partial \boldsymbol{v}_t}{\partial t} + \boldsymbol{v}_t\cdot\nabla\boldsymbol{v}_t - \frac{1}{\bar\rho_t}(\nabla\cdot\boldsymbol{H}_t)\boldsymbol{H}_t - \frac{1}{\bar\rho_t}(\nabla\times\boldsymbol{H}_t)\times\boldsymbol{H}_t + \frac{1}{\bar\rho_t}\partial_i\{\beta_t U_t U_t\} = 0\\
&\frac{\partial \boldsymbol{H}_t}{\partial t} - \boldsymbol{H}_t\cdot\nabla\boldsymbol{v}_t + \boldsymbol{v}_t\cdot\nabla\boldsymbol{H}_t + \boldsymbol{H}_t(\nabla\cdot\boldsymbol{v}_t) = 0\\
&\frac{\partial U_t}{\partial t} + \boldsymbol{v}_t\cdot\nabla U_t + (\gamma_t - 1)U_t\nabla\cdot\boldsymbol{v}_t = 0
\end{aligned}\tag{3.17}$$

while the conservation laws of mass and entropy are satisfied due to (3.11):

$$\frac{\partial\bar\rho_t}{\partial t} + \nabla\bar\rho_t\boldsymbol{v}_t = 0 \tag{3.18}$$

$$\frac{\partial\bar\sigma_t}{\partial t} + \nabla\bar\sigma_t\boldsymbol{v}_t = 0. \tag{3.19}$$

If no magnetic monopole exists at $t = 0$, i.e.

$$\nabla\cdot\boldsymbol{H}(x) = 0 \qquad \text{for any } x \in M \text{ at } t = 0 \tag{3.20}$$

and if, for the internal energy $U_t = U_{\Psi_t}$,

$$\bar U_t(x) = U_t(x) \qquad \text{for any } x \in M \text{ at } t = 0 \tag{3.21}$$

then equation set (3.19) becomes the following equation set representing the motion of an isentropic MHD fluid:

$$\begin{aligned}
&\frac{\partial \boldsymbol{v}_t}{\partial t} + \boldsymbol{v}_t\cdot\nabla\boldsymbol{v}_t - \frac{1}{\rho_t}(\nabla\times\boldsymbol{H}_t)\times\boldsymbol{H}_t + \frac{1}{\rho_t}\nabla P_t = 0\\
&\frac{\partial \boldsymbol{H}_t}{\partial t} - \nabla\times(\boldsymbol{v}_t\times\boldsymbol{H}_t) = 0\\
&\frac{\partial U_t}{\partial t} + \boldsymbol{v}_t\cdot\nabla U_t + \frac{P_t}{\rho_t}\nabla\cdot\boldsymbol{v}_t = 0
\end{aligned}\tag{3.22}$$

where the pressure $P_t(x)$ satisfies the following condition:

$$P_t(x) = \rho_t(x)\left\{\rho_t(x)\frac{\partial U_t}{\partial \rho_t}(x) + \sigma_t(x)\frac{\partial U_t}{\partial \sigma_t}(x)\right\} \tag{3.23}$$

which is consistent with the first law of thermodynamics. It should be noted that the condition of (3.20) and the evolution equation of the magnetic field in (3.17) lead to $\nabla \cdot \boldsymbol{H} = 0$ for $t > 0$, and that (3.21), (3.18) and (3.19) result in $\bar{U}_t(x) = U_t(x)$ for $t > 0$.

When w_t is independent of the entropy density σ_t, equation set (3.22) represents the equation of motion for a barotropic fluid, and the function U_t no longer represents the internal energy. In this case, the conservation law of entropy (3.19) does not have to be satisfied, while (3.18) is still valid.

When the following condition is satisfied, (3.22) becomes the equation of motion for an isentropic fluid:

$$\boldsymbol{H}(x) = 0 \qquad \text{for any } x \in M \text{ at } t = 0. \tag{3.24}$$

To obtain the *non-homogeneous* I-fluid motion from the covariant derivatives of (3.15), the weighted Hodge decomposition theorem (introduced by Marsden (1976)) is first used to define an operator P_Ψ which orthogonally projects $V|_\Psi^- = (v^i(x)\partial_i, H_i(x)\,\mathrm{d}x^i, U(x))|_\Psi^- \in T_\Psi\mathcal{S}(M)$ onto $T_\Psi\mathcal{I}_\nu(M)$ for $\Psi = (\phi, h) \in \mathcal{I}_\nu(M)$, namely

$$P_\Psi[V|_\Psi^-] = \left(\left\{v^i(x) - \frac{1}{\rho_\phi(x)}\partial^i\rho_\phi(x)\theta(x)\right\}\partial_i, H_i(x)\,\mathrm{d}x^i\right)\Bigg|_\Psi^- \tag{3.25}$$

where the C^∞ function $\theta : M \to \mathbb{R}$ is defined such that

$$\partial_i\left(v^i(x) - \frac{1}{\rho_\phi(x)}\partial^i\rho_\phi(x)\theta(x)\right) = 0 \qquad \text{for any } x \in M.$$

Next, the operator of (3.25) projects the covariant derivatives $\tilde{\nabla}_{V_1|_\Phi^-}V_2|_\Phi^-$ of (3.15) on $T_\Phi\mathcal{I}(M)$ onto $T_\Phi\mathcal{I}_\nu(M)$ at $\Phi \in \mathcal{I}_\nu(M) \subset \mathcal{I}_*(M)$ for $V_1, V_2 \in i_\nu(M)$ (for $i_\nu(M)$: the Lie algebra of $\mathcal{I}_\nu(M)$), which enables us to obtain the covariant derivatives $\hat{\nabla}_{V|_\Phi^-}V|_\Phi^- \in T_\Phi\mathcal{I}_\nu(M)$ as

$$\hat{\nabla}_{V|_\Phi^-}V|_\Phi^- = P_\Phi[\tilde{\nabla}_{V|_\Phi^-}V|_\Phi^-].$$

The resultant geodesic on $\mathcal{I}_\nu(M)$ then represents the equation of motion for a non-homogeneous I-MHD fluid, which is, in fact, the equation of motion for a non-homogeneous I-fluid under the initial condition of (3.24) (see Ono (1994) for an alternative formulation of the motion of a homogeneous I-fluid or I-MHD fluid).

Finally, it should be noted that using the covariant derivatives of (3.15) gives the curvature tensor $\tilde{R}$ of $\mathcal{I}_*(M)$:

$$\tilde{R}(V_1|_\Psi^-, V_2|_\Psi^-) = [\tilde{\nabla}_{V_1|_\Psi^-}, \tilde{\nabla}_{V_2|_\Psi^-}] - \tilde{\nabla}_{[V_1|_\Psi^-, V_2|_\Psi^-]}. \tag{3.26}$$

3.2. A simple example

To apply the presented method, let us consider as a simple example the motion of an isentropic gas with no magnetic field present. The equation of motion becomes

$$\begin{aligned}&\frac{\partial v_t}{\partial t}+v_t\cdot\nabla v_t+\frac{1}{\rho_t}\nabla P_t=0\\&\frac{\partial U_t}{\partial t}+v_t\cdot\nabla U_t+\frac{P_t}{\rho_t}\nabla\cdot v_t=0\end{aligned}\tag{3.27}$$

where the pressure $P_t(x)$ is defined as

$$P_t(x)=\rho_t(x)\left\{\rho_t(x)\frac{\partial U_t}{\partial\rho_t}(x)+\sigma_t(x)\frac{\partial U_t}{\partial\sigma_t}(x)\right\}.\tag{3.28}$$

As shown in (3.22), the equation of motion can be obtained as the geodesic on the following Lie group:

$$\mathcal{I}_{H=0}(M)=\mathcal{D}(M)\times\mathcal{F}(M).\tag{3.29}$$

For $N\geqslant 2$, the equations of motion in (3.27) have a steady-state solution set $(v_t^j(x)\partial_j,U_t(x))=(v^j(x)\partial_j,U)$ satisfying the following incompressibility condition:

$$\begin{aligned}&\nabla\cdot v=0\\&\rho_t(x)=\rho(=\text{constant}).\end{aligned}\tag{3.30}$$

For example, when M is a two-dimensional flat torus, the following state solution satisfies (3.31):

$$\begin{aligned}&v^1(x)=k_2\sin(k_1x+k_2y)\\&v^2(x)=-k_1\sin(k_1x+k_2y)\end{aligned}\tag{3.32}$$

and

$$\rho_t(x)=\rho$$

which is also a solution to the evolution equation of an I-fluid, being unstable because of its non-positive sectional curvatures on $\mathcal{D}_v^+(M)$ (see Arnold 1978). Now, considering the flow of a gas with heat capacity ratio $\gamma=C_p/C_v$, we have

$$p=C\rho^\gamma\qquad U=\frac{C}{(\gamma-1)}\rho^{\gamma-1}$$

where $1<\gamma=C_p/C_v<2$. By using (3.15), $V|_{\bar\Phi_t}=(v_t,\rho_t)|_{\bar\Phi_t}$ has a sectional curvature of $\mathcal{I}_{H=0}(M)$ with $W|_{\bar\Phi_t}=(w_t,\varsigma_t)|_{\bar\Phi_t}$ as

$$\begin{aligned}\langle\!\langle W_t|_{\bar\Phi_t},\bar R(W_t|_{\bar\Phi_t},V_t|_{\bar\Phi_t})V_t|_{\bar\Phi_t}\rangle\!\rangle|_{\bar\Phi_t}=\int_M\mu_x\big\{&\tfrac{1}{2}(\gamma-1)C\rho^\gamma(\nabla\cdot w_t)^2\\&+\rho\{(\gamma-1)\nabla\varsigma_t+w_t\cdot\nabla v\}^2-\rho(w_t\cdot\nabla v)^2\big\}\end{aligned}\tag{3.33}$$

which is positive when $w_t^i(x)=c_tv^i(x)$ for a scalar c_t.

On the other hand, from the condition of (2.41), ς_t develops as

$$\frac{\partial\varsigma_t}{\partial t}+v\cdot\nabla\varsigma_t+C\rho^{\gamma-1}(\nabla\cdot w_t)=0.\tag{3.34}$$

4. Conclusions

The theorem presented showed that Lie–Poisson systems for semi-direct product groups can be described in terms of Riemannian geometry. Although the theorem was only applied to a MHD system having isentropic flow, it is expected to have applications to other Lie–Poisson systems so as to enable their nonlinear phenomena to be investigated without using a perturbation method.

When M is an N-dimensional flat torus, the method employed for MHD fluid motion can easily be described in terms of Fourier components. However, use of ordinary coordinate systems to calculate the covariant derivatives of (3.15) and curvature tensor of (3.26) is considered more beneficial than applying the Fourier series, while the reverse is true for I-fluid or I-MHD fluid motion (see Arnold, 1966, 1978, Ono 1994).

It should be realized that the method utilized is quite different from the Jacobi–Maupertuis method, which can also describe Hamiltonian systems in terms of Riemannian geometry. (For further details consult Abraham and Marsden (1978), and for its application to N-body problems using numerical simulation see Pettini (1992).) Their method, however, requires changing an initial parameter to another time parameter in order to obtain the equation of motion as the geodesic equation, whereas no such parameter switch is needed by the utilized method.

It is expected that the methodology described will enable conventional canonical Hamiltonian systems to be represented in terms of Riemannian geometry. Towards this end, such an extension is being developed.

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